Geometry Definitions, Postulates, and Theorems

Chapter 2: Reasoning and Proof

Section 2.1: Use Inductive Reasoning

Standards: 1.0 Students demonstrate understanding by identifying and giving examples of undefined terms, axioms, theorems, and inductive and deductive reasoning. 3.0 Students construct and judge the validity of a logical argument and give counterexamples to disprove a statement.

**Conjecture** - An unproven statement that is based on observations and patterns.

Ex. What’s next?  
- **Visual Pattern**  
  a) ![Visual Pattern](image)
  
  b) ![Visual Pattern](image)
  
  c) ![Visual Pattern](image)

- **Number Pattern**  
  a) 2, 5, 8, 11, 14  
  add 3 to preceding number
  
  b) 5, 7, 11, 17, 25, 35  
  add 2, then 4, then 6, ...

**Inductive Reasoning** - Process of looking for patterns and making conjectures.

Ex. The product of an odd number and an even number is **even**.

What is the pattern observed?

\[
\begin{align*}
3(8) &= 24 \\
6(5) &= 30 \\
11(24) &= 264 \\
7(10) &= 70
\end{align*}
\]

Ex. The sum of an odd number and an even number is **odd**.

\[
\begin{align*}
1 + 4 &= 5 \\
8 + 37 &= 45 \\
21 + 10 &= 31 \\
13 + 18 &= 31
\end{align*}
\]

**Counterexample** – An example that shows that a conjecture is false.

Ex. The difference between two positive numbers is always positive. Find a counterexample. (there are many correct answers)

\[3 - 9 = -6\]
Section 2.2: Analyze Conditional Statements

**Standards: 3.0** Students construct and judge the validity of a logical argument and give counterexamples to disprove a statement.

**Conditional Statement** – A logical statement that has two parts, a hypothesis and a conclusion. A conditional statement is best when written in IF . . . , THEN . . . form.

**Hypothesis** – The "IF" part of a conditional statement.

**Conclusion** – The "THEN" part of a conditional statement.

Ex. Rewrite the conditional statements in if-then form:

a) It is time for dinner if it is 6pm. **If it is 6pm, then it is time for dinner.**

b) All monkeys have tails. **If an animal is a monkey, then it has a tail.**

*Conditional statements can be true or false. Decide whether the statement is true or false. If false, provide a counterexample.

Ex. If a number is odd, then it is divisible by 3.

**Converse** – A statement formed by switching the hypothesis and conclusion of a conditional statement. NOTE: A converse is not always a true statement.

Ex. Statement: If you see lightning, then you hear thunder.

Converse: **If you hear thunder, then you see lightning.**

**Negation** – Writing the opposite or negative of the statement.

Ex. a) \( m \angle A = 35^\circ \), \( m \angle A \neq 35^\circ \)  b) \( \angle D \) is obtuse, \( \angle D \) is not obtuse

**Inverse** – The opposite of the hypothesis **AND** the opposite of the conclusion of a conditional statement.

Ex. Statement: If an animal is a dog, then it has four legs.

Inverse: **If an animal is not a dog, then it does not have four legs.**

**Contrapositive** – The opposite of the hypothesis **AND** the opposite of the conclusion of a converse.

Ex. Statement: If an animal is a fish, then it can swim.

Contrapositive: **If an animal cannot swim, then it is not a fish.**
**Equivalent Statements** - Statements that are both true or both false.

Ex. a) Statement: If $m \angle A = 30^\circ$, then $\angle A$ is acute.

Contrapositive: If $\angle A$ is not acute, then $m \angle A \neq 30^\circ$. 

b) Inverse: If $m \angle A \neq 30^\circ$, then $\angle A$ is not acute.

Converse: If $\angle A$ is acute, then $m \angle A = 30^\circ$.

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**Perpendicular Lines** – IF two lines intersect to form a right angle, THEN the lines are perpendicular.

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**Biconditional Statement** – A statement that contains the phrase "if and only if." This is equivalent to a conditional statement and its converse.

Biconditional Statement: $x + 3 = 5$ if and only if $x = 2$.

Conditional: If $x + 3 = 5$, then $x = 2$. \[ \text{T/F} \]

Converse: If $x = 2$, then $x + 3 = 5$. \[ \text{T/F} \]

The biconditional statement is true since the conditional statement is true and the converse is true.

Biconditional Statement: $x^2 = 4x$ if and only if $x = 4$.

Conditional: If $x^2 = 4x$, then $x = 4$. \[ \text{T/F} \]

Converse: If $x = 4$, then $x^2 = 4x$. \[ \text{T/F} \]

The biconditional statement is false since the conditional statement is false.
Section 2.3: Apply Deductive Reasoning

**Standards:** 1.0 Students demonstrate understanding by identifying and giving examples of undefined terms, axioms, theorems, and inductive and deductive reasoning. 3.0 Students construct and judge the validity of a logical argument and give counterexamples to disprove a statement.

**Deductive Reasoning** relies on facts, definitions and accepted properties in logical order to write a logical argument.

**Inductive Reasoning** examples and patterns are used to form a conjecture.

*Two types of Deductive Reasoning:*

**Law of Detachment** – IF $p \rightarrow q$ is true, AND $p$ is true, THEN $q$ is true.

Ex. If $m \angle ABC < 90^\circ$, then $\angle ABC$ is acute.

Given that $m \angle DEF = 42^\circ$,
we can conclude that $\angle DEF$ is acute.

**Law of Syllogism** – IF $p \rightarrow q$ is true, AND $q \rightarrow r$ is true, THEN $p \rightarrow r$ is true.

*Like the transitive property (cut out the middle man)*

Ex. If $m \angle ABC = 45^\circ$, then $m \angle ABC < 90^\circ$.

If $m \angle ABC < 90^\circ$, then $\angle ABC$ is acute.

If $m \angle ABC = 45^\circ$, then $\angle ABC$ is acute.

Ex. Decide whether deductive or inductive reasoning is being used in the following argument.

Everyday Mrs. Maya reads off the homework answers, then goes over the wrong ones on the board. Today she has read off the homework answers, so you can conclude that she will go over the wrong answers on the board.

**Inductive** Why did you decide so? **Based on patterns.**

Ex. **Law of Syllogism**

$p \rightarrow q$ If you are a student, then you have lots of homework.

$q \rightarrow r$ If you have lots of homework, then you have little social life.

$p \rightarrow r$ If you are a student, then you have little social life.
Section 2.3 Extension

**Symbolic Notation:** $p$ represents the hypotheses
$q$ represents the conclusion

conditional statement (if-then): $p \rightarrow q$

converse $q \rightarrow p$

negation of $p$ $\sim p$

inverse $\sim p \rightarrow \sim q$

contrapositive $\sim q \rightarrow \sim p$

biconditional $p \leftrightarrow q$ (means $p \rightarrow q$ and $q \rightarrow p$)

Ex. Use $p$ and $q$ to write the symbolic statement in words.

$p$: A number is divisible by 3.
$q$: A number is divisible by 6.

$p \rightarrow q$ If a number is divisible by 3, then it is divisible by 6.

$q \rightarrow p$ If a number is divisible by 6, then it is divisible by 3.

$\sim p$ A number is not divisible by 3.

$\sim p \rightarrow \sim q$ If a number is not divisible by 3, then it is not divisible by 6.

$\sim q \rightarrow \sim p$ If a number is not divisible by 6, then it is not divisible by 3.

$p \leftrightarrow q$ A number is divisible by 3 if and only if a number is divisible by 6.
Section 2.4: Use Postulates and Diagrams

Standards: 1.0 Students demonstrate understanding by identifying and giving examples of undefined terms, axioms, theorems, and inductive and deductive reasoning.

**Postulate 5** – IF you are given any two points, THEN there exists exactly one line.

**Postulate 6** – IF you are given a line, THEN it contains at least two points.

**Postulate 7** – IF two lines intersect, THEN their intersection is exactly one point.

**Postulate 8** – IF you are given any three noncollinear points, THEN there exists exactly one plane.

**Postulate 9** – IF you are given a plane, THEN it contains at least three noncollinear points.

**Postulate 10** – IF two points lie in a plane, THEN the line containing them also lies in the plane.

**Postulate 11** – IF two planes intersect, THEN the intersection is a line.

**Line Perpendicular to a Plane** – A line that intersects a plane at a point and makes a right angle with the plane.

*Ex. State the postulate that verifies the truth of the statement: V and T lie on line n.*

*Write out the postulate.*

**Postulate 5:** Through any two points, there is exactly one line.
Section 2.5: Reason Using Properties from Algebra

Standards:
1.0 Students demonstrate understanding by identifying and giving examples of undefined terms, axioms, theorems, and inductive and deductive reasoning.
3.0 Students construct and judge the validity of a logical argument and give counterexamples to disprove a statement.

Addition Property of Equality – \( a = b \)  
\[ x - 2 = 10 \]  
\[ x = 12 \quad \text{add 2 to both sides} \]

Subtraction Property of Equality – \( a = b \)  
\[ 5 + n = 8 \]  
\[ n = 3 \quad \text{subtract 5 from both sides} \]

Multiplication Property of Equality – \( a = b \)  
\[ x/2 = 4 \]  
\[ x = 8 \quad \text{multiply both sides by 2} \]

Division Property of Equality – \( a = b \)  
\[ 4y = 20 \]  
\[ y = 5 \quad \text{Divide both sides by 4} \]

Substitution Property of Equality – \( a = b \)  
If \( \angle A = 100^\circ \) and \( m\angle A = m\angle B \), then  
\[ m\angle B = 100^\circ \]

Distributive Property – \( a(b + c) = ab + ac \) OR \( ab + ac = a(b + c) \)  
\[ 2(x + 4) = 2x + 8 \]  
\[ a(b - c) = ab - ac \] OR \( ab - ac = a(b - c) \]

or \( 5x^2 - 3x = x(5x - 3) \)

Properties of Equality for Real Numbers, Segments and for Angles

**Reflexive**

Real Numbers
IF \( a \) is a real #  
THEN \( a = a \)
\[ 10 = 10 \]
\[ x = x \]

Segments
IF Segment \( AB \)  
THEN \( AB = AB \)

Angles
IF Angle \( A \)  
THEN \( m\angle A = m\angle A \)

**Symmetric**

IF \( a = b \)  
THEN \( b = a \)

IF \( AB = CD \)  
THEN \( CD = AB \)

IF \( m\angle A = m\angle B \)  
THEN \( m\angle B = m\angle A \)

If \( 10 = x \), then \( x = 10 \).

**Transitive**

IF \( a = b \)  
AND \( b = c \),  
THEN \( a = c \).

IF \( AB = CD \)  
AND \( CD = EF \),  
THEN \( AB = EF \).

IF \( m\angle A = m\angle B \),  
AND \( m\angle B = m\angle C \),  
THEN \( m\angle A = m\angle C \).

*cut out the middle man*  
If \( a = x + 3 \),  
and \( x + 3 = 7 \),  
then \( a = 7 \).
**Ex. Algebraic Proof –** Solve the equation $3x + 12 = 8x - 18$ and state the reason for each step.

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $3x + 12 = 8x - 18$</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. $12 = 5x - 18$</td>
<td>2. Subtraction Prop. of =</td>
</tr>
<tr>
<td>3. $30 = 5x$</td>
<td>3. Addition Prop. of =</td>
</tr>
<tr>
<td>4. $6 = x$</td>
<td>4. Division Prop. of =</td>
</tr>
<tr>
<td>5. $x = 6$</td>
<td>5. Symmetric Prop.</td>
</tr>
</tbody>
</table>

**Ex. Geometric Proof –** Show the perimeter of $\triangle ABC$ is equal to the perimeter of $\triangle ADC$.

<table>
<thead>
<tr>
<th>Statements</th>
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</tr>
</thead>
<tbody>
<tr>
<td>1. $AD \cong AB; DC \cong BC$</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. $AC \cong AC$</td>
<td>2. Reflexive</td>
</tr>
<tr>
<td>3. $\overbrace{AB + BC + CA} = \text{Perimeter of } \triangle ABC$</td>
<td>3. Definition of Perimeter</td>
</tr>
<tr>
<td>4. $\overbrace{AD + DC + CA} = \text{Perimeter of } \triangle ADC$</td>
<td>4. Substitution</td>
</tr>
<tr>
<td>5. Perimeter of $\triangle ADC =$ Perimeter of $\triangle ABC$</td>
<td>5. Substitution</td>
</tr>
</tbody>
</table>

**Ex. Use the property to complete the statement.**

Transitive Property: If $YZ \cong DB$ and $\overbrace{DB =JK}$, then $YZ = JK$. 
Section 2.6 – Prove Statements about Segments & Angles

Standards: 1.0 Students demonstrate understanding by identifying and giving examples of undefined terms, axioms, theorems, and inductive and deductive reasoning. 2.0 Students write geometric proofs, including proofs by contradiction.

**Proof** – A logical argument that shows a statement is true.

**Two-Column Proof** – Numbered statements and corresponding reasons that show an argument in a logical order.

<table>
<thead>
<tr>
<th>Statements</th>
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</tr>
</thead>
<tbody>
<tr>
<td>1)</td>
<td>1)</td>
</tr>
<tr>
<td>2)</td>
<td>2)</td>
</tr>
<tr>
<td>etc)</td>
<td>etc)</td>
</tr>
</tbody>
</table>

**Theorem** – A statement that can be proven. Once you have proven a theorem, you can use the theorem as a reason in other proofs.

***Theorem 2.1  Congruence of Segments is Reflexive***
GIVEN: Segment $AB$,
THEN $\overline{AB} \cong \overline{AB}$.

***Theorem 2.1  Congruence of Segments is Symmetric***
IF $\overline{AB} \cong \overline{CD}$,
THEN $\overline{CD} \cong \overline{AB}$.

***Theorem 2.1  Congruence of Segments is Transitive***
IF $\overline{AB} \cong \overline{CD}$,
AND $\overline{CD} \cong \overline{EF}$,
THEN $\overline{AB} \cong \overline{EF}$.

Ex. Given: $\overline{PQ} \cong \overline{XY}$  
Prove: $\overline{XY} = \overline{PQ}$

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $\overline{PQ} \cong \overline{XY}$</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. $\overline{PQ} = \overline{XY}$</td>
<td>2. Def. of $\cong$ Segments</td>
</tr>
<tr>
<td>3. $\overline{XY} = \overline{PQ}$</td>
<td>3. Symmetric Property</td>
</tr>
</tbody>
</table>

(over)
Ex. Given: $Q$ is the midpoint of $PR$.  
Prove: $PQ = \frac{1}{2} PR$

1. $Q$ is the midpoint of $PQ$  
2. $PQ = QR$  
3. $PR = PQ + QR$  
4. $PR = PQ + PQ$, $PR = 2PQ$  
5. $\frac{1}{2} PR = PQ$  
6. $PQ = \frac{1}{2} PR$

1. Given  
2. Definition of Midpoint  
4. Substitution  
5. Division Prop. of $=$  

Ex. Given: $EF = GH$  
Prove: $EG = FH$

1. $EF = GH$  
2. $FG = FG$  
3. $(EF + FG) = (GH + FG)$  
4. $EG = EF + FG$, $FH = GH + FG$  
5. $EG = FH$

1. Given  
2. Reflexive  
3. Add. Prop. of $=$  
4. Segment Addition Postulate  
5. Substitution

Ex. Given: $BH \cong AH$, $BH = 12$  
Prove: $AH = 12$

1. $BH \cong AH$, $BH = 12$  
2. $BH = AH$  
3. $12 = AH$  
4. $AH = 12$

1. Given  
2. Definition of Congruent Segments  
3. Substitution  
4. Symmetric Prop.

Ex. In the diagram, if $AB \cong BC$ and $BC \cong CD$, find $BC$.

$3x - 1 = 2x + 3$  
$x - 1 = 3$  
$x = 4$  
$BC = 2x + 3$  
$BC = 2(4) + 3$  
$BC = 11$
***Theorem 2.2 Congruence of Angles is Reflexive
GIVEN: Angle A,
THEN \( \angle A \cong \angle A \).

***Theorem 2.2 Congruence of Angles is Symmetric
IF \( \angle A \cong \angle B \),
THEN \( \angle B \cong \angle A \).

***Theorem 2.2 Congruence of Angles is Transitive
IF \( \angle A \cong \angle B \),
AND \( \angle B \cong \angle C \),
THEN \( \angle A \cong \angle C \).

Ex. Proof of Transitive Property of Congruence

Given: \( \angle A \cong \angle B \), \( \angle B \cong \angle C \)
Prove: \( \angle A \cong \angle C \)

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) ( \angle A \cong \angle B ), ( \angle B \cong \angle C )</td>
<td>1) given</td>
</tr>
<tr>
<td>2) ( m \angle A = m \angle B ), ( m \angle B = m \angle C )</td>
<td>2) Def. of ( \cong ) angles</td>
</tr>
<tr>
<td>3) ( m \angle A = m \angle C )</td>
<td>3) Transitive Property</td>
</tr>
<tr>
<td>4) ( \angle A \cong \angle C )</td>
<td>4) Def. of ( \cong ) angles</td>
</tr>
</tbody>
</table>

Ex. Given: \( \angle 1 \cong \angle 2 \), \( \angle 3 \cong \angle 4 \), \( \angle 2 \cong \angle 3 \)
Prove: \( \angle 1 \cong \angle 4 \)

1. \( \angle 1 \cong \angle 2 \), \( \angle 3 \cong \angle 4 \), \( \angle 2 \cong \angle 3 \)
2. \( \angle 1 \cong \angle 4 \)
3. \( \angle 1 \cong \angle 4 \)

Ex. Given: \( m \angle 1 = 63^\circ \), \( \angle 1 \cong \angle 3 \), \( \angle 3 \cong \angle 4 \)
Prove: \( m \angle 4 = 63^\circ \)

1. \( m \angle 1 = 63^\circ \), \( \angle 1 \cong \angle 3 \), \( \angle 3 \cong \angle 4 \)
2. \( \angle 1 \cong \angle 4 \)
3. \( m \angle 1 = m \angle 4 \)
4. \( 63^\circ = m \angle 4 \)
5. \( m \angle 4 = 63^\circ \)
Section 2.7 – Prove Angle Pair Relationships

**Standards:**

1.0 Students demonstrate understanding by identifying and giving examples of undefined terms, axioms, theorems, and inductive and deductive reasoning.

2.0 Students write geometric proofs, including proofs by contradiction.

***Theorem 2.3  Right Angle Congruence Theorem***

All right angles are congruent.

Ex. Given: \( \angle DAB \) and \( \angle ABC \) are right angles, \( \angle ABC \cong \angle BCD \)
Prove: \( \angle DAB \cong \angle BCD \)

1. \( \angle DAB \) and \( \angle ABC \) are right angles
2. \( \angle DAB \cong \angle ABC \)
3. \( \angle DAB \cong \angle BCD \)

**Theorem 2.4  Congruent Supplements Theorem***

If two angles are supplementary to the same angle (or \( \cong \) angles),
then they are congruent.

**Theorem 2.5  Congruent Complements Theorem***

If two angles are complementary to the same angle (or \( \cong \) angles),
then they are congruent.

Ex. Given: \( m\angle 1 = 24^\circ \), \( m\angle 3 = 24^\circ \), \( \angle 1 \) and \( \angle 2 \) are complementary,
\( \angle 3 \) and \( \angle 4 \) are complementary
Prove: \( \angle 2 \cong \angle 4 \)

1. \( m\angle 1 = 24^\circ \), \( m\angle 3 = 24^\circ \)
2. \( \angle 1 \) and \( \angle 2 \) are complementary,
\( \angle 3 \) and \( \angle 4 \) are complementary
3. \( m\angle 1 = m\angle 3 \)
4. \( \angle 2 \cong \angle 4 \)

***Linear Pair Postulate*** – If two angles form a linear pair,
then they are supplementary

\( \angle 1 + \angle 2 = 180^\circ \)
**Theorem 2.6   Vertical Angles Theorem**
Vertical angles are congruent.

Ex.  Given:  \( \angle 1 \) and \( \angle 2 \) are a linear pair, \( \angle 2 \) and \( \angle 3 \) are a linear pair
Prove:  \( \angle 1 \cong \angle 3 \)

<table>
<thead>
<tr>
<th>1. ( \angle 1 ) and ( \angle 2 ) are a linear pair</th>
<th>1. Given</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \angle 2 ) and ( \angle 3 ) are a linear pair</td>
<td>2. Linear Pair Postulate</td>
</tr>
<tr>
<td>( \angle 1 ) &amp; ( \angle 2 ) are supp.</td>
<td>3. Congruent Supp. Theorem</td>
</tr>
<tr>
<td>( \angle 2 ) &amp; ( \angle 3 ) are supp.</td>
<td>3. Congruent Supp. Theorem</td>
</tr>
<tr>
<td>( \angle 1 \cong \angle 3 )</td>
<td>3. Congruent Supp. Theorem</td>
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</table>

Ex.  See diagram

a) \( m\angle AQB = 90^\circ \) (vertical angles)
b) \( m\angle CQA = 90^\circ \) (linear pair)
c) If \( m\angle CQD = 31^\circ \), then \( m\angle EQF = 31 \)
d) If \( m\angle BQG = 125^\circ \), then \( m\angle CQF = 125 \)
e) \( m\angle AQB + m\angle GQF + m\angle EQG = 180 \)
f) If \( m\angle EQF = 38^\circ \), then \( m\angle BQC = 52 \)